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The AdS_4 Gravitational Perturbation and Supersymmetry

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Abstract Gravitational perturbation in AdS_4, decomposed into axial and polar perturbations that have different parities. After imposing geometric perturbation in to the spherical, symmetric and static metric of AdS_4, we can calculate the possible spectrum of frequencies around the static configuration. This spectrum relates to the boundary conditions imposed at spatial infinity ($r = \infty$) and will be implied by the wave function's behavior at $x = \frac{\pi}{2}$ or $y = 1$. At the end we write down the first order raising and lowering operators from the second order Schrödinger equation to study the supersymmetry model and check whether we have supersymmetric partners or no. After finding the H_1 and the partner H_2, we check the supersymmetry. At the end, we find the potentials to confirm the shape invariance condition and also the supercharges and their commutative relations are studied.

Keywords AdS/CFT correspondence · Supersymmetry · AdS_4 space-time · Shape invariance

1 Introduction

By using the AdS/CFT correspondence [1–3], we have found a good framework to study the relation between two distinct physical laws; one, on the boundary and the other in the interior space of the anti-deSitter. As we know the AdS is the best candidate for the quantum theory of gravity that all the holographic principles are satisfied in it. The interior space consists of gravity and an extra dimension and on the other hand, the boundary is gravityless with a lower dimension. These two physics are apparently different but thanks to the holographic principle, they are equivalent. On the boundary, there are interacting particles

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similar to quark and gluons of standard physics in the realm but with more than three colors. Conformal field theory is dominant on the boundary and the string theory is the best approach for the interior physics. The gravity is generated by the interactions of fields and particles on the boundary of AdS and the extra dimension is a fact caused by the thickness of particles on the boundary, it means that, how much the particle is thinner, the corresponding particle in the interior space is nearer to the boundary. Since the equivalence is hold in this space then we expect, each configuration in the interior has an equivalent on the boundary.

We work on AdS_4/CFT_3, because it has an special characteristic related to the gravitational perturbations. As we told, there are two types of axial and polar perturbations that have different parity and the polar perturbations are complementary to the axial ones [4]. Axial perturbation corresponds to the vector sector or sound channel and polar perturbations relates to the scalar sector or shear channel in the dictionary of the AdS/CFT correspondence.

These fluctuations have general forms that are given by [4], consist of four functions of each parameter (t, r, θ, φ). The $\delta g_{\mu\nu}$ is parameterized in terms of these parameters and ω as a sum of both axial and polar perturbation terms in this space-time. This allowed spectrum of frequencies motivate us to find the special constraints of stability. we are also interested in checking the presence of supersymmetry, its partners and supercharges. Solving the Einstein equation for the space-time with negative cosmological constant (Λ), represent the following metric for the AdS space-time that is written in terms of the spherical coordinates and implies a symmetric and static manifold,

$$dS^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $f(r)$ is a function of parameters of the space-time. As we know, different backgrounds may have different parameters such as; cosmological constant, mass, charge and etc. Here, we consider the simplest case (without mass and charge) but a negatively curved space-time. Then for $f(r)$ we have,

$$f(r) = 1 - \frac{\Lambda}{3}r^2. \tag{2}$$

After imposing gravitational perturbations to the metric, two classes of perturbations (Axial and Polar) appear with different potentials and wave functions correspond to the Regge–Wheeler an Zerilli quantities,

$$g_{\mu,\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \tag{3}$$

where $g_{\mu,\nu}$ is the perturbed metric of the AdS_4, and $g^{(0)}_{\mu\nu}$ represent the unperturbed metric and $\delta g_{\mu\nu}$ term correspond to the sum of axial and polar perturbation imposed to the metric. For considering these perturbations we should refer to the Regge–Wheeler [6] and use the quantities given for the AdS_4 black holes. It is important to point out that here we have used the analogy with four dimensional black holes and put the mass $(M = 0)$ in the Regge-Wheeler and Zerilli potentials. For the first class (axial), we have the following equation for the potential that is defined for AdS_4 black holes [6],

$$V_{eff} = f(r)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right), \tag{4}$$

and for the polar perturbation of the AdS_4 black holes, we have the Zerilli potential [7],

$$V_{eff} = \frac{f(r)}{[(l-1)^2(l+2)^2 r + 6M]}(l(l+1)(l-1)^2(l+2)^2 - 24M^2\Lambda + \frac{6M}{r}(l-1)^2(l+2)^2) + h(r)), \tag{5}$$

where $h(r)$ is,

$$h(r) = \frac{f(r)}{[(l-1)^2(l+2)^2 r + 6M]}\left(\frac{36M^4}{r^2}(l-1)(l+2) + \frac{72M^3}{r^3}\right). \tag{6}$$

Here we consider the simplest space ($M = 0$) in the absence of black holes. So, in this case, two potentials will be same.

So, by introducing the new coordinates as $dr_\star = \frac{dr}{f(r)}$ and also change of variable $x = \sqrt{\frac{-\Lambda}{3}} r_\star$, two effective potentials will be as [4],

$$V(x) = \frac{l(l+1)}{\sin^2 x}, \tag{7}$$

the boundary conditions imposed to the wave function at $r = \infty$, then the mentioned change of variable can help us to study the allowed frequencies of oscillation at $x = \frac{\pi}{2}$. In AdS_4, there are two types of dualities as energy-momentum/cotton tensor duality on the boundary and electric/magnetic duality in the bulk that should be equivalent to each other according to the holographic principle dominant in the AdS space. It means that energy-momentum/cotton tensor duality on the boundary is a holographic representation of the electric/magnetic duality in the bulk. I. Bakas studied these dualities for AdS_4 space-time [4] and also AdS_4 black holes [5]. Here we note that these dualities lead us to discuss the supersymmetric partners for special boundary conditions to check the presence of supersymmetry, shape invariance and also to find the supercharges. Here we want to apply geometric perturbations on the static metric of AdS_4 and find the possible spectrum of frequencies to fluctuate around static configurations (Sect. 2). In Sect. 3 we study the supersymmetric model and find the supersymmetric partners, supercharges and their commutative relations with the Hamiltonian.

2 Possible Spectrum of Frequencies

Now we are going to write the corresponding Schrödinger equation. This help us to obtain the behavior of wave function and investigate the spectra frequencies and stability of AdS_4 for some condition. As we know the effective potential (7) can be obtained by gravity perturbation and duality information. So, Schrödinger equation associated with the potential given by (7) can be solved by Gegenpour polynomials. In that case we have following equation,

$$\left(-\frac{d^2}{dx^2} + \frac{l(l+1)}{\sin^2 x}\right)\Psi(x) = \Omega^2\Psi(x). \tag{8}$$

With the definition of variable $\sin x = y$ and $\Psi(y) = U(y)P^{\lambda}_{n,m}(y)$, one can obtain the corresponding equation as, following;

$$(1-y^2)P''^{(\lambda)}_{n,m}(y) + \left[\frac{2u'}{u}(1-y^2) - y\right]P'^{(\lambda)}_{n,m} + \left[\frac{u''}{u}(1-y^2) - y\frac{u'}{u} + \Omega^2 - \frac{l(l+1)}{y^2}\right]P^{(\lambda)}_{n,m}(y) = 0, \tag{9}$$

where n, m are real parameters. In order to obtain wave function and spectrum frequencies we need to introduce the Gegenpour equation which is given by [8],

$$(1-y^2)P''^{(\lambda)}_{n,m}(y) - 2(\lambda+1)yP'^{(\lambda)}_{n,m}(y) + \left(n(2\lambda+n+1) - \frac{m(2\lambda+m)}{1-y^2}\right)P^{(\lambda)}_{n,m}(y) = 0. \tag{10}$$

Now, by comparing (9) and (10), we obtain the wave function in terms of y and x, so we have,

$$\Psi(y) = C_{n,m}(1-y^2)^{\frac{2\lambda+1}{4}}P^{\lambda}_{n,m}(y), \tag{11}$$

and

$$\Psi(y) = C_{n,m}(\cos x)^{\frac{2\lambda+1}{2}}P^{\lambda}_{n,m}(y), \tag{12}$$

where $C_{n,m}$ will be obtained with normalization condition. For obtaining spectrum of frequencies we need to compare the third term of (9) and (10) with each other. So, the spectrum frequencies can be obtained by the following equation,

$$\Omega^2 = \frac{l(l+1)}{y^2} - \frac{2y^2b(2b-1) + m(2\lambda+m)}{(1-y^2)} + n(2\lambda+n+1), \tag{13}$$

where $b = \frac{2\lambda+1}{4}$.

As we told before, after imposing geometric perturbations to the static, symmetric and spherical metric of AdS_4, we have an allowed spectrum of fluctuations around the static configuration that is directly related to the boundary conditions imposed at spatial infinity. Then we should study the behavior of the wave functions on the conformal boundary of the space. This boundary is located at $r=\infty$, then it is easier to work with $x=\frac{\pi}{2}$ or $y=1$. Now we apply the boundary conditions [4] which are as follow,

$$0 < r < \infty, \qquad 0 < x < \frac{\pi}{2}, \qquad 0 < y < 1. \tag{14}$$

So, the allowed spectra frequency and the stability situation of the system is given by the following relation in terms of real parameters n, m,

$$\Omega^2 = l(l+1) + n(2+n-m), \quad \lambda = -m \pm \frac{1}{2}. \tag{15}$$

This stability condition and the corresponding Schrödinger equation lead us to discuss the supersymmetry.

3 Supersymmetric Version of AdS_4 Space-Time

As we know from [7–10], we can factorize the second order equation in terms of the first order equation, it means that the Schrödinger equation will be factorized to give the first order generators and the corresponding Hamiltonian H_1 and H_2. Also the shape invariance condition [8–13] lead us to have supersymmetry partners which is important in several branches of physics. In that case we factorize (9) in terms of first order equation with respect to n and m. First, we factorized the corresponding equation with respect to n,

$$\begin{aligned} A_n^+(y) &= (1-y^2)\frac{d}{dy} - (2\lambda + n)y, \\ A_n^-(y) &= -(1-y^2)\frac{d}{dy} - ny, \end{aligned} \tag{16}$$

where $y = \sin x$. So, we have the following equation

$$\begin{aligned} A_n^+(x) &= \cos x \frac{d}{dx} - (2\lambda + n)\sin x, \\ A_n^-(x) &= -\cos x \frac{d}{dx} - n\sin x. \end{aligned} \tag{17}$$

Also we factorized the second order equation (9) with respect to m,

$$\begin{aligned} A_m^+(y) &= \sqrt{(1-y^2)}\frac{d}{dy} + \frac{(m-1)y}{\sqrt{(1-y^2)}}, \\ A_m^-(y) &= \sqrt{(1-y^2)}\frac{d}{dy} + \frac{(2\lambda+n)}{\sqrt{(1-y^2)}}y, \end{aligned} \tag{18}$$

and

$$\begin{aligned} A_m^+(x) &= \frac{d}{dx} + (m-1)\tan x, \\ A_m^-(x) &= -\frac{d}{dx} + (2\lambda + m)\tan x. \end{aligned} \tag{19}$$

In order to discuss the supersymmetry version of AdS_4 we need to introduce the $H_1 = A_n^+(x)A_n^-(x)$ and $H_2 = A_n^-(x)A_n^+(x)$. The supersymmetry condition implies to have $H_1 = H_2$. Now we are going to apply the supersymmetry condition for the factorized equation with respect n and m. In the first case, if we have $\lambda = -n$ we will arrive at relation $A_n^+(x)A_n^-(x) = A_n^-(x)A_n^+(x)$ $(H_1 = H_2)$. In the second case always satisfied $H_1 = H_2$. All these help us to obtain the Q, Q^+ of generators of supersymmetry algebra and more partner potential and superpotential. By introducing $H_1 = A_n^+(x)A_n^-(x)$ and $H_2 = A_n^-(x)A_n^+(x)$ and using the analogy with $H_1 = \frac{p^2}{2\mu} + V_1$ and $H_2 = \frac{p^2}{2\mu} + V_2$ we can check the shape invariance. Here we should stress that supersymmetry in quantum mechanic is based upon the factorization method in the framework of shape invariance The shape invariance conditions constraint V_1 and V_2 be same as $V_2 = V_1 + c$. In the special case $\lambda = \frac{-1-n}{2}$ for the factorized equation with respect to n the corresponding Hamiltonian H_1 as follow,

$$H_1 = -\cos^2 x \frac{d^2}{dx^2} - n, \tag{20}$$

where $V_1 = -n$.

Also the Hamiltonian H_2,

$$H_2 = -\cos^2 x \frac{d^2}{dx^2} + (2\lambda + 1), \tag{21}$$

where $V_2 = (2\lambda + 1)$.

The shape invariance condition implies to have $V_1(-n) = v_2(2\lambda + 1)$. Now we can find H_1 and H_2 in terms of the parameter m. By introducing $H_1 = A_m^+(x)A_m^-(x)$ and $H_2 = A_m^-(x)A_m^+(x)$ and using the analogy with $H_1 = \frac{p^2}{2\mu} + V_1$ and $H_2 = \frac{p^2}{2\mu} + V_2$ we can check the shape invariance. In this case we have following equation for H_1 and H_2, also

$$H_1 = -\frac{d^2}{dx^2} - (2\lambda + 1)(1 + \tan^2 x), \tag{22}$$

where $V_1 = (2\lambda + 1)(1 + \tan^2 x)$.

Also the corresponding Hamiltonian H_2 is

$$H_2 = -\frac{d^2}{dx^2} - (m - 1)(1 + \tan^2 x), \tag{23}$$

where $V_2 = -(m - 1)(1 + \tan^2 x)$.

With $V_1(2\lambda + 1) = V_2(1 - m)$ we have shape invariance. Now we are sure about the presence of supersymmetry. Then, we can write down the supercharges in terms of the both parameters n and m.

$$Q = \begin{pmatrix} 0 & 0 \\ -\cos x \frac{d}{dx} - n \sin x & 0 \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} 0 & \cos x \frac{d}{dx} - (2\lambda + n)\sin x \\ 0 & 0 \end{pmatrix}, \tag{24}$$

Now we discuss the second case which is factorized with respect to m and the corresponding supercharges are as,

$$Q = \begin{pmatrix} 0 & 0 \\ -\frac{d}{dx} + (2\lambda + m)\tan x & 0 \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} 0 & \frac{d}{dx} + (m - 1)\tan x \\ 0 & 0 \end{pmatrix}. \tag{25}$$

Relations between the generators of algebra;

$$\begin{gathered} [H, Q] = [H, Q^+] = 0, \\ \{Q, Q^+\} = \{H, \{Q, Q\} = \{Q^+, Q^+\} = 0, \end{gathered} \tag{26}$$

where the corresponding Hamiltonian H is

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \tag{27}$$

since we have

$$[H, Q] = \begin{pmatrix} 0 & 0 \\ H_2 A - A H_1 & 0 \end{pmatrix}. \tag{28}$$

Then the condition $H_2 A = A H_1$ is obtained to satisfy the relation $[H, Q] = 0$.

4 Conclusion

AdS_4 space-time is the best background for studying gravitational perturbations, because it has two distinct classes of perturbations that are defined in the general forms by Regge–Wheeler in [4]. These perturbed terms for both axial and polar classes of perturbations are defined for AdS_4 black holes and here we have used the analogy of these two backgrounds and put the mass zero in Regge–Wheeler and Zerilli potentials. In this letter, we have studied the allowed spectrum of frequencies for fluctuation around the static configuration. This spectra can be achieved by considering the behavior of wave function on the boundary of AdS. Then, we write out the Schrödinger equations for both types of axial and polar perturbations that led to the same equation. We wrote the resultant wave function in terms of polynomial functions and by comparing with [7], the possible spectrum is achieved. Since, these frequencies are directly related to the boundary conditions imposed at spatial infinity, then the spectra should satisfy some special constraints. The presence of supersymmetry and the resultant shape invariance was another goal that was checked by defining the Hamiltonian in terms of raising and lowering generators and applying the equality $H_1 = H_2$. This can ensure the presence of supersymmetric partners and also the shape invariance ($V_1 = V_2$) as a result of supersymmetry. At the end we find the supercharges and their algebra. This process can be applied for the Minkowski space too.

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